Investigations about Monads

Dissertation

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1 Introduction

Eilenberg and Mac Lane invented categories in the 1940s as a way of relating systems of algebraic structures and systems of topological spaces in algebraic topology. The spread of applications led to a general theory. Category theory not only yields a relation between these fields but is also a generalisation of concepts which occur in many different branches of mathematics. Historically category theory arose in algebraic topology as a way to explain in what sense the passages from geometry to algebra in that field are *natural* in the sense of reflecting underlying geometric reality rather than particular representations in that reality. The theory of categories and functors abstracts the concepts *object* and *map* from the underlying mathematical fields, for example, from algebra or topology, to investigate which statements can be proved in such an abstract structure. Then these statements will be true in all those mathematical fields which can be expressed by means of the language provided by category theory. Many different mathematical fields may be interpreted as categories and then the techniques and theorems of this theory may be applied to these fields. It often turns out that certain proofs in different branches use *similar* methods. With this language it is possible to express these *similarities* in exact terms. Category theory also provides, in terms of functors and natural transformations, methods which can carry propositions from one field of mathematics to corresponding propositions in another field, preserving their validity.

In recent years category theory has also become interesting for computer scientists. Nowadays it cannot be avoided in discussions of the semantics of programming languages, for example, the polymorphic lambda calculus which underlies ML (a functional programming language). In Computer Science, category theory can help, for example, to construct higher order functions in a functional programming language, e.g., Miranda[1], as done in [Gri88]. Exception handling is an important part of these studies. The question of composing failure trapping functions can be handled by use of monads, a structure in category theory.

Monads have now become so important that some computer scientists wish a programming language to provide constructors for monads. This, however, leads to the question as to whether there is a canonical way of constructing a new monad from given ones. In other words; is it possible to compose two monads so that the composite is again a monad, analogous to composing functions in an ordinary programming language?

Although the origin for this dissertation, to examine the as yet unsolved question of monad compositions, lies in Computer Science, this document contains only the mathematical apparatus which is independent from the computational background. It will give an example of two composable monads and some general (but obviously too strong conditions) for composing two monads.

[1] Miranda is a trademark of Research Software Ltd.

2 Basic Definitions

Usually, in mathematics the notation of symbols and formulae are abbreviated so that the proper meaning can be deduced from the context. However, although this kind of abbreviation is customary and for the working mathematician even necessary, for someone who starts learning a new field it is often difficult to recognise the background which lies behind such abbreviations. That is why in this paper the symbolism nearly everywhere carries more information than would be necessary for someone who is familiar with categories, functors and natural transformations.

Since this dissertation does not aim to be an introduction to category theory we only give brief and perhaps imprecise definitions of the basic concepts category, functor and natural transformation to clarify the notation we use here.

Definition 1 *A* category $\mathbf{C}$ *consists of a class of* objects A, B, C, $\ldots$, *and a disjoint family of sets* $\{\mathbf{C}(A,B)\}$, *one set for each ordered pair* $\langle A,B\rangle$ *of objects. An element* $f \in \mathbf{C}(A,B)$ *is called a* morphism *or an* arrow *from* A *to* B, *written* $f : A \to B$ *or* $A \xrightarrow{f} B$. A *is called* domain *and* B codomain *of* f. *For arrows* $f \in \mathbf{C}(A,B)$, $g \in \mathbf{C}(B,C)$ *there is a* composed arrow $g \circ f \in \mathbf{C}(A,C)$. *This composition is associative. Whenever* $h \circ g$ *and* $g \circ f$ *are defined, the equality* $(h \circ g) \circ f = h \circ (g \circ f)$ *holds. For each object* A *there is an* identity arrow $1_A \in \mathbf{C}(A,A)$, *and for all objects* A, B *and all morhisms* $f \in \mathbf{C}(A,B)$ $1_B \circ f = f = f \circ 1_A$ *is valid.*

Definition 2 *A* functor $S : \mathbf{C} \to \mathbf{D}$ *from a category* $\mathbf{C}$ *to a category* $\mathbf{D}$ *is a mapping which assigns to each object* A *in* $\mathbf{C}$ *an object* SA *in* $\mathbf{D}$, *and to each arrow* $f \in \mathbf{C}(A,B)$ *an arrow* $Sf \in \mathbf{D}(SA,SB)$ *so that:*

1. $S(g \circ f) = Sg \circ Sf$ *whenever* $g \circ f$ *is defined in* $\mathbf{C}$
2. $S(1_A) = 1_{SA}$ *for all objects* A *in* $\mathbf{C}$

If the source and target category are the same, the functor is called an endofunctor. *The* identity endofunctor *of a category* $\mathbf{C}$ *is the functor which just returns its argument and is denoted by* $I_{\mathbf{C}}$ *or simply* I.

If we regard categories as universal algebras with partially defined operations (as composition is), the concept of a functor becomes more familiar since then it is a homomorphism between two categories.

Analogous to normal functions, functors can be composed by obvious rules: If $S : \mathbf{A} \to \mathbf{B}$ and $T : \mathbf{B} \to \mathbf{C}$ so $(T \circ S)A = T(SA)$ and $(T \circ S)f = T(Sf)$. It is easy to prove that $T \circ S$ is a functor. We write S^2 as short-hand for $S \circ S$.

Definition 3 *Given a pair of functors* $S, T : \mathbf{A} \to \mathbf{B}$. *A* natural transformation $\tau : S \dot{\to} T$ *is a mapping which assigns to each object* A *of* $\mathbf{A}$ *an arrow*

$\tau_A \in \mathbf{B}(SA, TA)$ of $\mathbf{B}$ in such a way that each arrow $f : A \to A'$ in $\mathbf{A}$ yields a diagram

$$\begin{tikzcd} A \arrow[d, "f"] & SA \arrow[r, "\tau_A"] \arrow[d, "Sf"] & TA \arrow[d, "Tf"] \\ A' & SA' \arrow[r, "\tau_{A'}"] & TA' \end{tikzcd}$$

which is commutative, i.e., $\tau_{A'} \circ Sf = Tf \circ \tau_A$.

Definition 4 *A* monad $\mathcal{M} = \langle S, \eta, \mu \rangle$ *in a category $\mathbf{A}$ consists of a functor $S : \mathbf{A} \to \mathbf{A}$ and two natural transformations $\eta : I \dot{\to} S$ and $\mu : S^2 \dot{\to} S$ which make for all objects A of $\mathbf{A}$ the following diagrams commute*

$$\begin{tikzcd} S^3A \arrow[r, "\mu_{SA}"] \arrow[d, "S(\mu_A)"] & S^2A \arrow[d, "\mu_A"] \\ S^2A \arrow[r, "\mu_A"] & SA \end{tikzcd}$$

and

$$\begin{tikzcd} I(SA) \arrow[r, "\eta_{SA}"] \arrow[dr, "1_{SA}"'] & S^2A \arrow[d, "\mu_A"] & S(IA) \arrow[l, "S(\eta_A)"'] \arrow[dl, "1_{SA}"] \\ & SA & \end{tikzcd}$$

Here the symbol I denotes the identity functor of $\mathbf{A}$. The commutativity of these diagrams is the same as to say the equations

$$\mu_A \circ S(\mu_A) = \mu_A \circ \mu_{SA} \tag{1}$$

$$\mu_A \circ S(\eta_A) = \mu_A \circ \eta_{SA} = 1_{SA} \tag{2}$$

hold.

Since the first diagram expresses a kind of associativity law, and the second and third are left and right unit laws for the monad, μ is referred to as the *multiplication* of the monad, while η is called the *unit*.

3 Notations

Here is a short summary of the conventions used in this paper:
If $f : A \to B$ then $f\,a = f(a)$ and if $f : A \to C^B$ then $f\,a\,b = (f\,a)\,b = (f(a))(b)$. If $f : A \to B$ and $g : B \to C$ then $(g \circ f)\,a = g(f(a))$. Parentheses are necessary here, otherwise we would have $g \circ f\,a = g \circ (f\,a)$. Bold face letters refer to categories, e.g., $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, … Capital letters at the beginning of the alphabet denote objects of a category. The capital letters I, J, P, R, S, T are used for functors. Lower case letters like f, g, h, … are used to denote category arrows. Lower case letters in general refer to elements of sets. Greek letters denote natural transformations.

4 Example

This section will give one particular example of a composed monad which proves the existence of at least one monad which can be regarded as a composite of two other monads. Although the involved monads are fairly simple, the example reveals first problems for a general method of monad composition as references to this section will show.

4.1 Given monads

The category **Set** is a category with objects all (small) sets and arrows all (total) functions between sets. In **Set** we fix two sets X and Y. We define the Monads $\mathcal{M} = \langle S, \eta, \mu \rangle$ and $\mathcal{M}' = \langle S', \eta', \mu' \rangle$ by the following equations.

Let I be the identity functor on **Set** and $h : A \to B$ be any function from set A to set B. Then

$$\begin{aligned} SA &= A^X \\ Sh &= h_* : A^X \to B^X \\ &\quad \text{where } h_* \, g = h \circ g \text{ for } g \in A^X \end{aligned}$$

The unit $\eta : I \dot{\to} S$ has components $\eta_A : A \to A^X$ and is defined by

$$\eta_A \, a \, x = a.$$

The multiplication $\mu : S^2 \dot{\to} S$ with components $\mu_A : (A^X)^X \to A^X$ is given by

$$\mu_A \, h \, x = h \, x \, x.$$

For the monad $\mathcal{M}' = \langle S', \eta', \mu' \rangle$ let

$$\begin{aligned} S'A &= A + Y \\ S'h &= h + 1_Y. \end{aligned}$$

Let $i_A : A \to A + Y$ and $j_A : Y \to A + Y$ be the inclusion functions for $A + Y$. Since the set Y is fixed as second component of the disjoint union in this section, we can omit an index Y at the inclusion functions which would just repeat that fact. Then the natural transformation

$$\begin{aligned} &\eta' : I \dot{\to} S' \\ &\eta'_A : A \to A + Y \end{aligned}$$

is defined by

$$\eta'_A = i_A$$

For the multiplication of the second monad we have

$$\begin{aligned} &\mu' : S'^2 \dot{\to} S' \\ &\mu'_A : (A + Y) + Y \to A + Y \end{aligned}$$

where

$$\begin{aligned} \mu'_A(i_{(A+Y)}(i_A(a))) &= i_A(a) \\ \mu'_A(i_{(A+Y)}(j_A(y))) &= j_A(y) \\ \mu'_A(j_{(A+Y)}(y')) &= j_A(y') \end{aligned}$$

4.2 Question

Can we assign a monad structure to $S \circ S'$?

This question is fairly special since our aim is only to find a way (in some sense canonical) to construct a new monad from two given ones. However, this does not necessarily require the functor of the new monad to be the usual composition of the given functors. Despite this fact we shall see in this section that the above question has a positive answer.

Let $\hat{S} = S \circ S'$. Then we have

$$\begin{aligned} \hat{S}A &= (A+Y)^X \\ \hat{S}h &= (h+1_Y)_* : (A+Y)^X \to (B+Y)^X \end{aligned}$$

In order to define a monad $\hat{\mathcal{M}} = \langle \hat{S}, \hat{\eta}, \hat{\mu} \rangle$, we define $\hat{\eta} : I \dot{\to} \hat{S}$ and $\hat{\mu} : \hat{S}^2 \dot{\to} \hat{S}$ to be the following mappings.

It is

$$\begin{aligned} &\hat{\eta}_A : A \to (A+Y)^X, \\ &(\hat{\eta}_A\, a) : X \to A+Y \end{aligned}$$

and $\hat{\eta}$ is defined by

$$\hat{\eta}_A\, a\, x = i_A(a). \tag{3}$$

As we shall see later, this definition of $\hat{\eta}$ in the case $\hat{S} = S \circ S'$ can easily be obtained from η and η'. Here, however, the definition comes from a look at the definitions of η and η' and the mathematical *experience* to build the right definition for $\hat{\eta}$ rather than a clear algorithm to *construct* this natural transformation.

Now, let $h : X \to (A+Y)^X + Y$. We define $(\hat{\mu}_A\, h) : X \to A+Y$ as follows. If $h\, x = i_{\hat{S}A}(g)$ for $g \in (A+Y)^X$ then

$$(\hat{\mu}_A\, h)\, x = g\, x. \tag{4}$$

If $h\, x = j_{\hat{S}A}(y)$ then we define

$$(\hat{\mu}_A\, h)\, x = j_A(y). \tag{5}$$

4.3 Answer

As stated above, we shall show that this structure is indeed a monad. Since in the last section we only gave definitions of $\hat{\mathcal{M}}$, we have to verify that its components have the correct properties.

Proposition 1 *$\hat{S} = S \circ S'$ is an endofunctor of* **Set**.

Proof: Since composition of functors gives a new functor, and both S and S' are endofunctors of **Set**, so is $\hat{S}$.

□

First of all we must show that $\hat{\eta}$ and $\hat{\mu}$ are natural transformations.

Proposition 2 *$\hat{\eta} : I \dot{\to} \hat{S}$ is a natural transformation.*

Proof: To prove $\hat{\eta}$ to be natural, we must show that the following diagram commutes for all objects A, B and all arrows $f \in \mathbf{Set}(A, B)$.

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\hat{\eta}_A"] \arrow[d, "f"] & (A+Y)^X \arrow[d, "(f+1_Y)_*"] \\ B \arrow[r, "\hat{\eta}_B"] & (B+Y)^X \end{tikzcd}$$

Therefore we must prove $(f + 1_Y)_* \circ \hat{\eta}_A = \hat{\eta}_B \circ f$. For arbitrary $a \in A$, $x \in X$ let $L = (\hat{S}f \circ \hat{\eta}_A)\, a\, x$ and $R = (\hat{\eta}_B \circ f)\, a\, x$. Then

$$\begin{aligned} L &= (f + 1_Y)_*(\hat{\eta}_A\, a)\, x \\ &= ((f + 1_Y) \circ (\hat{\eta}_A\, a))\, x \\ &= (f + 1_Y)(\hat{\eta}_A\, a\, x) \\ &\stackrel{(3)}{=} (f + 1_Y)(i_A(a)) \\ &= i_B(f\, a) \end{aligned} \tag{6}$$

$$\begin{aligned} R &= (\hat{\eta}_B \circ f)\, a\, x \\ &= \hat{\eta}_B\, (f\, a)\, x \\ &\stackrel{(3)}{=} i_B(fa) \end{aligned} \tag{7}$$

Comparing (6) and (7) proves $\hat{\eta}$ to be natural.

$\square$

Proposition 3 *$\hat{\mu} : \hat{S}^2 \dot{\to} \hat{S}$ is a natural transformation.*

Proof: To prove $\hat{\mu}$ to be natural, we must show that the following diagram commutes for all objects A, B and all arrows $f \in \mathbf{Set}(A, B)$.

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \qquad \begin{tikzcd} ((A+Y)^X+Y)^X \arrow[r, "\hat{\mu}_A"] \arrow[d, "((f+1_Y)_*+1_Y)_*"] & (A+Y)^X \arrow[d, "(f+1_Y)_*"] \\ ((B+Y)^X+Y)^X \arrow[r, "\hat{\mu}_B"] & (B+Y)^X \end{tikzcd}$$

Therefore we must prove $(f + 1_Y)_* \circ \hat{\mu}_A = \hat{\mu}_B \circ ((f + 1_Y)_* + 1_Y)_*$.

For arbitrary $h \in ((A + Y)^X + Y)^X$ and $x \in X$ let

$$L = ((f + 1_Y)_* \circ \hat{\mu}_A)\, h\, x$$

and

$$R = (\hat{\mu}_B \circ ((f + 1_Y)_* + 1_Y)_*)\, h\, x.$$

Then

$$\begin{aligned}
L &= (f + 1_Y)_*(\hat{\mu}_A\, h)\, x \\
&= ((f + 1_Y) \circ (\hat{\mu}_A\, h))\, x \\
&= (f + 1_Y)(\hat{\mu}_A\, h\, x).
\end{aligned}$$

Let $L' = h\,x$. There are two cases

1. $h\,x = i_{\hat{S}A}(g)$ for some $g \in (A + Y)^X = \hat{S}A$
   Then $\hat{\mu}_A\, h\, x \overset{(4)}{=} g\, x$.

$$L = (f + 1_Y)(g\, x) \tag{8}$$

2. $h\,x = j_{\hat{S}A}(y)$ for some $y \in Y$
   Then $\hat{\mu}_A\, h\, x \overset{(5)}{=} j_A(y)$.

$$\begin{aligned}
L &= (f + 1_Y)(j_A(y)) \\
&= j_B(y)
\end{aligned} \tag{9}$$

For the right-hand side we deduce:

$$\begin{aligned}
R &= (\hat{\mu}_B \circ ((f + 1_Y)_* + 1_Y)_*)\, h\, x \\
&= \hat{\mu}_B\, (((f + 1_Y)_* + 1_Y)_*\, h)\, x \\
&= \hat{\mu}_B\, (((f + 1_Y)_* + 1_Y) \circ h)\, x
\end{aligned}$$

Let $R' = (((f + 1_Y)_* + 1_Y) \circ h)\, x = ((f + 1_Y)_* + 1_Y)(h\, x)$. There are again two cases.

1. $h\,x = i_{\hat{S}A}(g)$ for some $g \in (A + Y)^X = \hat{S}A$

$$\begin{aligned}
R' &= ((f + 1_Y)_* + 1_Y)(i_{\hat{S}A}(g)) \\
&= i_{\hat{S}B}((f + 1_Y)_*\, g) \\
&= i_{\hat{S}B}((f + 1_Y) \circ g)
\end{aligned}$$

   Since $(f + 1_Y) \circ g \in (B + Y)^X$ we can conclude from (4)

$$\begin{aligned}
R &= ((f + 1_Y) \circ g)\, x \\
&= (f + 1_Y)(g\, x).
\end{aligned} \tag{10}$$

2. $h\,x = j_{\hat{S}A}(y)$ for some $y \in Y$

$$\begin{aligned}
R' &= ((f + 1_Y)_* + 1_Y)(j_{\hat{S}A}(y)) \\
&= j_{\hat{S}B}(y)
\end{aligned}$$

   Hence

$$R \overset{(5)}{=} j_B(y). \tag{11}$$

Comparing (8) and (9) with (10) and (11) respectively proves $\hat{\mu}$ to be natural.

$\square$

Now, it only remains to show that $\hat{\mathcal{M}} = \langle \hat{S}, \hat{\eta}, \hat{\mu} \rangle$ it indeed a monad. The following diagrams must commute for all objects A.
(Unit property)

$$\begin{tikzcd}
(A+Y)^X \arrow[r, "\eta_{\hat{S}A}"] \arrow[dr, "1_{\hat{S}A}"'] & ((A+Y)^X+Y)^X \arrow[d, "\hat{\mu}_A"] & (A+Y)^X \arrow[l, "\hat{S}(\hat{\eta}_A)"'] \arrow[dl, "1_{\hat{S}A}"] \\
 & (A+Y)^X &
\end{tikzcd}$$

and
(Associativity)

$$\begin{tikzcd}
(((A+Y)^X+Y)^X+Y)^X \arrow[r, "\hat{\mu}_{\hat{S}A}"] \arrow[d, "\hat{S}(\hat{\mu}_A)"] & ((A+Y)^X+Y)^X \arrow[d, "\hat{\mu}_A"] \\
((A+Y)^X+Y)^X \arrow[r, "\hat{\mu}_A"] & (A+Y)^X
\end{tikzcd}$$

Proposition 4 *The first diagram commutes.*

Proof: We shall prove the commutativity for the left-hand and right-hand part of the above diagram separately.

First look at the left-hand triangle. Here we must show the identity

$$\hat{\mu}_A \circ \hat{\eta}_{\hat{S}A} = 1_{\hat{S}A}.$$

Let $h \in (A+Y)^X$ and $x \in X$. By definition of $\hat{\eta}$, it is

$$\hat{\eta}_{\hat{S}A}\, h\, x = i_{\hat{S}A}(h).$$

Then follows from (4) that

$$\begin{aligned}
(\hat{\mu}_A \circ \hat{\eta}_{\hat{S}A})\, h\, x &= \hat{\mu}_A(\hat{\eta}_{\hat{S}A}\, h)\, x \\
&\overset{(4)}{=} h\, x
\end{aligned}$$

Hence $\hat{\mu}_A \circ \hat{\eta}_{\hat{S}A} = 1_{\hat{S}A}$.

To prove the corresponding identity, i.e., $\hat{\mu}_A \circ \hat{S}(\hat{\eta}_A) = 1_{\hat{S}A}$, for the right-hand triangle to be valid, let $h \in (A+Y)^X$, $x \in X$, $R = (\hat{\mu}_A \circ \hat{S}(\hat{\eta}_A))\, h\, x$ and $R' = \hat{S}(\hat{\eta}_A)\, h\, x$. In order to use the definition of $\hat{\mu}$, we derive

$$\begin{aligned}
R' &= ((\hat{\eta}_A + 1_Y)_*\, h)\, x \\
&= ((\hat{\eta}_A + 1_Y) \circ h)\, x \\
&= (\hat{\eta}_A + 1_Y)(h\, x).
\end{aligned}$$

This leads us to two cases, which we treat in the following way.

1. $h\,x = i_A(a)$ for some $a \in A$
   Then $R' = i_{\hat{S}A}(\hat{\eta}_A\, a)$, and by definition of $\hat{\mu}$ and $\hat{\eta}$

   $$R \overset{(4)}{=} \hat{\eta}\, a\, x \overset{(3)}{=} i_A(a) = h\, x.$$

2. $h\,x = j_A(y)$ for some $y \in Y$
   Then $R' = j_{\hat{S}A}(y)$, and by definition of $\hat{\mu}$

   $$R \overset{(5)}{=} j_A(y) = h\, x.$$

In both cases we got the right result which verifies $\hat{\mu}_A \circ \hat{S}(\hat{\eta}_A) = 1_{\hat{S}A}$. Thus the proposition is proved.

$\square$

Proposition 5 *The second diagram, which states the associativity, is commutative.*

Proof: It is to be shown that the following equation holds.

$$\hat{\mu}_A \circ \hat{S}(\hat{\mu}_A) = \hat{\mu}_A \circ \hat{\mu}_{\hat{S}A}$$

Let $h \in (((A+Y)^X + Y)^X + Y)^X$ and $x \in X$. Then deduce for the left-hand side

$$\begin{aligned} L &:= (\hat{\mu}_A \circ \hat{S}(\hat{\mu}_A))\, h\, x \\ &= \hat{\mu}_A\, ((\hat{\mu}_A + 1_Y)_*\, h)\, x. \end{aligned} \tag{12}$$

In order to apply $\hat{\mu}_A$, look at the expression

$$\begin{aligned} L' &:= ((\hat{\mu}_A + 1_Y)_*\, h)\, x \\ &= ((\hat{\mu}_A + 1_Y) \circ h)\, x \\ &= (\hat{\mu}_A + 1_Y)(h\, x). \end{aligned}$$

This forces us to investigate two cases.

1. $h\,x = i_{\hat{S}^2 A}(g)$ for some $g \in ((A+Y)^X + Y)^X$
   Then $L' = i_{\hat{S}A}(\hat{\mu}_A\, g)$ with $(\hat{\mu}_A\, g) \in (A+Y)^X$ and thus by definition of $\hat{\mu}$ for the leftmost $\hat{\mu}_A$ in (12)

   $$L \overset{(4)}{=} \hat{\mu}_A\, g\, x.$$

2. $h\,x = j_{\hat{S}^2 A}(y)$ for some $y \in Y$
   Then $L' = j_{\hat{S}A}(y)$, and by definition of $\hat{\mu}$

   $$L \overset{(5)}{=} j_A(y).$$

For the right-hand side deduce

$$\begin{aligned} R &:= (\hat{\mu}_A \circ \hat{\mu}_{\hat{S}A})\, h\, x \\ &= \hat{\mu}_A\, (\hat{\mu}_{\hat{S}A}\, h)\, x. \end{aligned}$$

Let $R' = \hat{\mu}_{\hat{S}A}\, h\, x$. To apply $\hat{\mu}_A$, look at the following two cases.

1. $h\, x = i_{\hat{S}^2 A}(g)$ for some $g \in ((A+Y)^X + Y)^X$
   Then $R' = (\hat{\mu}_{\hat{S}A}\, h)\, x \overset{(4)}{=} g\, x$. Hence $\hat{\mu}_{\hat{S}A}\, h = g$, and therefore

   $$R = \hat{\mu}_A\, g\, x.$$

2. $h\, x = j_{\hat{S}^2 A}(y)$ for some $y \in Y$
   Then $R' \overset{(5)}{=} j_{\hat{S}A}(y)$, and by definition of $\hat{\mu}$

   $$R \overset{(5)}{=} j_A(y).$$

Compare the results for the left-hand and right-hand side. This proves the asserted proposition.

$\square$

4.4 Conclusion

This chapter not only proved $\hat{\mathcal{M}}$ to be a monad but also gave an example of the *existence* of a monad which is a composite of two other monads.

5 Further Definitions

In order to make formulae more readable for the following investigations, we introduce compositions for natural transformations. As we shall see, each of these composites defined next will be a natural transformation. In other words: if we build expressions from natural transformations (and functors) by following compositions, we need not prove that the resulting map is natural.

Firstly, we give the notation for identity natural transformations.

Definition 5 *Let $I : \mathbf{A} \to \mathbf{A}$ be the identity functor of a category $\mathbf{A}$. Then the mapping $\iota : I \dot{\to} I$ with components $\iota_A = 1_A : A \to A$ is called the identity natural transformation from I to I.*
If $T : \mathbf{A} \to \mathbf{B}$ is a functor from $\mathbf{A}$ to $\mathbf{B}$ then we call the mapping $\iota_T : T \dot{\to} T$ defined by $(\iota_T)_A = \iota_{(TA)}$ the identity natural transformation of T to itself.

To be more precise, I should carry a subscript to show of which category it is the identity functor. However, this would overload the notation and is in fact unnecessary. So we denote by I the identity functor of any category and by ι the identity natural transformation from the identity functor to itself without mentioning the category involved. Only if more than one identity functor is in discussion will we choose a different notation.

Some authors, for example, Mac Lane in [ML71], write $R\tau$ and τR respectively for natural transformations with components $(R\tau)_A = R(\tau_A)$ and $(\tau R)_A = \tau_{RA}$ respectively. We shall clarify this in choosing another notation since we do not use functor symbols to denote the identity natural transformation of that functor to itself, and we also do not omit the (horizontal) composition sign.

All definitions and proofs are extracted from [ML71], adapted for this paper and also slightly changed. The definition of right and left composites are new concepts which cannot be found in [ML71]. These definitions will help to understand a notation which usually occurs in books about categories. We shall define horizontal composites in terms of right and left composites and so shorten the definition of Mac Lane.

5.1 Right and left composites of natural transformations with functors

Definition 6 *Let $T : \mathbf{A} \to \mathbf{B}$, $R, R' : \mathbf{B} \to \mathbf{C}$ and $P : \mathbf{C} \to \mathbf{D}$ be functors and $\varrho : R \dot{\to} R'$ be a natural transformation. Then we call $\varrho \triangleright T : R \circ T \dot{\to} R' \circ T$ defined by*

$$(\varrho \triangleright T)_A = \varrho_{(TA)} \qquad \text{for any object } A \text{ of } \mathbf{A}$$

the right composite *of ϱ with T. The mapping $P \triangleleft \varrho : P \circ R \dot{\to} P \circ R'$ with components*

$$(P \triangleleft \varrho)_B = P(\varrho_B) \qquad \text{for any object } B \text{ of } \mathbf{B}$$

is called the left composite *of ϱ with P.*

$\varrho \triangleright T$ and $P \triangleleft \varrho$ are natural transformations since ϱ is natural and P and T are functors.

Note that now ι_T is the same as $\iota \triangleright T$.

Proposition 6 *In addition to the notation of the last definition let $\pi : P \dot{\to} P'$ and $\tau : T \dot{\to} T'$ be natural transformations. Then the equalities*

$$\begin{array}{rcl} (\pi \triangleright R) \triangleright T & = & \pi \triangleright (R \circ T) \\ (P \circ R) \triangleleft \tau & = & P \triangleleft (R \triangleleft \tau) \\ (P \triangleleft \varrho) \triangleright T & = & P \triangleleft (\varrho \triangleright T). \end{array} \tag{13}$$

hold.

Proof: This can be proved by looking at the following derivation sequences. Firstly,

$$\begin{array}{rcl} ((\pi \triangleright R) \triangleright T)_A & = & (\pi \triangleright R)_{(TA)} \\ & = & \pi_{R(TA)} \\ & = & \pi_{(R \circ T)A} \\ & = & (\pi \triangleright (R \circ T))_A, \end{array}$$

secondly, derive

$$\begin{array}{rcl} (P \triangleleft (R \triangleleft \tau))_A & = & P((R \triangleleft \tau)_A) \\ & = & P(R(\tau_A)) \\ & = & (P \circ R)(\tau_A) \\ & = & ((P \circ R) \triangleleft \tau)_A \end{array}$$

and finally

$$\begin{array}{rcl} ((P \triangleleft \varrho) \triangleright T)_A & = & (P \triangleleft \varrho)_{(TA)} \\ & = & P(\varrho_{(TA)}) \\ & = & P((\varrho \triangleright T)_A) \\ & = & (P \triangleleft (\varrho \triangleright T))_A \end{array}$$

which ends the proof.

$\square$

These identities justify the common practice of omitting the $\triangleright$ and $\triangleleft$ signs, because then even parentheses can be avoided which makes formulae more readable, and expressions like πRT, $P\varrho T$ and $PR\tau$ cannot lead to ambiguity.

5.2 Vertical composite of natural transformations

Left and right composites are only auxiliary concepts. More important are the concepts of vertical and horizontal composites. We shall adopt these concepts from [ML71] but we shall use a slightly different notation. We write $\diamond$ instead of $\circ$ for horizontal composition to distinguish it from ordinary composition of arrows and functors.

Definition 7 *Given functors $T, T', T'' : \mathbf{A} \to \mathbf{B}$ and natural transformations $\tau : T \dot{\to} T'$ and $\tau' : T' \dot{\to} T''$. Then define the* vertical composite $\tau' \bullet \tau : T \dot{\to} T''$ *by*

$$(\tau' \bullet \tau)_A = \tau'_A \circ \tau_A \tag{14}$$

for an arbitrary object A of $\mathbf{A}$.

Proposition 7 *Using the notation of definition 7, it follows that $\tau' \bullet \tau$ is a natural transformation.*

Proof: Take an arbitrary arrow $f : A \to A'$ of $\mathbf{A}$ and consider the diagram

$$\begin{tikzcd}
TA \arrow[r, "\tau_A"] \arrow[d, "Tf"] & T'A \arrow[r, "\tau'_A"] \arrow[d, "T'f"] & T''A \arrow[d, "T''f"] \\
TA' \arrow[r, "\tau_{A'}"] & T'A' \arrow[r, "\tau'_{A'}"] & T''A'
\end{tikzcd}$$

By definition, the horizontal arrows from TA to $T''A$ and from TA' to $T''A'$ respectively are $(\tau' \bullet \tau)_A$ and $(\tau' \bullet \tau)_{A'}$ respectively. Since τ and τ' are natural both small squares are commutative. Hence the rectangle commutes, and so $\tau' \bullet \tau$ is natural.

□

This composition is associative, as can readily be proved by adding one more small square to the diagram above. Moreover it has for each functor T an identity, the identity natural transformation $\iota_T : T \dot{\to} T$ from T to itself. For an arbitrary object A of $\mathbf{A}$ we have

$$\begin{aligned}
(\iota_T \bullet \tau)_A &= (\iota_T)_A \circ \tau_A \\
&= \iota_{(TA)} \circ \tau_A \\
&= 1_{(TA)} \circ \tau_A \\
&= \tau_A
\end{aligned}$$

and

$$\begin{aligned}
(\tau \bullet \iota_T)_A &= \tau_A \circ (\iota_T)_A \\
&= \tau_A \circ \iota_{(TA)} \\
&= \tau_A \circ 1_{(TA)} \\
&= \tau_A
\end{aligned}$$

which proves the identity property of ι_T.

As we shall see in the next section, the following relations involving $\triangleright$, $\triangleleft$ and $\bullet$ will be useful to prove the associativity of the horizontal composition.

Proposition 8 *Let $T : \mathbf{A} \to \mathbf{B}$, $R, R', R'' : \mathbf{B} \to \mathbf{C}$, $P : \mathbf{C} \to \mathbf{D}$ be functors and $\varrho : R \dot{\to} R'$ and $\varrho' : R' \dot{\to} R''$ be natural transformations. Then following identities hold:*

$$\begin{aligned}
(\varrho' \bullet \varrho) \triangleright T &= (\varrho' \triangleright T) \bullet (\varrho \triangleright T) \\
P \triangleleft (\varrho' \bullet \varrho) &= (P \triangleleft \varrho') \bullet (P \triangleleft \varrho)
\end{aligned} \tag{15}$$

Proof: For any object A of $\mathbf{A}$ derive

$$\begin{aligned}
((\varrho' \bullet \varrho) \rhd T)_A &= (\varrho' \bullet \varrho)_{TA} \\
&\overset{(14)}{=} \varrho'_{TA} \circ \varrho_{TA} \\
&= (\varrho' \rhd T)_A \circ (\varrho \rhd T)_A \\
&\overset{(14)}{=} ((\varrho' \rhd T) \bullet (\varrho \rhd T))_A
\end{aligned}$$

and for any object B of $\mathbf{B}$:

$$\begin{aligned}
(P \lhd (\varrho' \bullet \varrho))_B &= P((\varrho' \bullet \varrho)_B) \\
&\overset{(14)}{=} P(\varrho'_B \circ \varrho_B) \\
&= P(\varrho'_B) \circ P(\varrho_B) \\
&= (P \lhd \varrho')_B \circ (P \lhd \varrho)_B \\
&\overset{(14)}{=} ((P \lhd \varrho') \bullet (P \lhd \varrho))_B .
\end{aligned}$$

This gives evidence for the validity of the proposition.

□

5.3 Horizontal composite of natural transformations

Now, we turn to another composition.

Definition 8 *For functors $T, T' : \mathbf{A} \to \mathbf{B}$, $R, R' : \mathbf{B} \to \mathbf{C}$ and natural transformations $\tau : T \dot{\to} T'$, $\varrho : R \dot{\to} R'$, the following diagram commutes since ϱ is natural and $\tau_A : TA \to T'A$ is an arrow of* $\mathbf{B}$

$$\begin{tikzcd}
R(TA) \arrow[r, "\varrho_{TA}"] \arrow[d, "R(\tau_A)"] & R'(TA) \arrow[d, "R'(\tau_A)"] \\
R(T'A) \arrow[r, "\varrho_{T'A}"] & R'(T'A)
\end{tikzcd}$$

Since this is true for all objects A, we omit the explicit appearance of the object. Then the diagram transforms to

$$\begin{tikzcd}
R \circ T \arrow[r, "\varrho \rhd T"] \arrow[d, "R \lhd \tau"] & R' \circ T \arrow[d, "R' \lhd \tau"] \\
R \circ T' \arrow[r, "\varrho \rhd T'"] & R' \circ T'
\end{tikzcd}$$

Define the horizontal composite $\varrho \diamond \tau : R \circ T \dot{\to} R' \circ T'$ *to be the diagonal arrow of the last diagram. Then we have*

$$\begin{aligned}
\varrho \diamond \tau &= (\varrho \rhd T') \bullet (R \lhd \tau) \\
&= (R' \lhd \tau) \bullet (\varrho \rhd T)
\end{aligned} \tag{16}$$

Proposition 9 *The mapping $\varrho \diamond \tau$ in the last definition is a natural transformation.*

To show $\varrho \diamond \tau : R \circ T \dot{\to} R' \circ T'$ natural, form

$$\begin{tikzcd}
(R \circ T)A \arrow[r, "(R \lhd \tau)_A"] \arrow[d, "(R \circ T)f"] & (R \circ T')A \arrow[r, "(\varrho \rhd T')_A"] \arrow[d, "(R \circ T')f"] & (R' \circ T')A \arrow[d, "(R' \circ T')f"] \\
(R \circ T)A' \arrow[r, "(R \lhd \tau)_{A'}"] & (R \circ T')A' \arrow[r, "(\varrho \rhd T')_{A'}"] & (R' \circ T')A'
\end{tikzcd}$$

for any arrow $f : A \to A'$ of $\mathbf{A}$. Horizontally, the composites by definition are $(\varrho \diamond \tau)_A$ and $(\varrho \diamond \tau)_{A'}$; the left-hand square commutes because τ is natural and R is a functor, while the right-hand square commutes because ϱ is natural and $T'f : T'A \to T'A'$ is an arrow. The commutativity of the outside of the diagram states that $\varrho \diamond \tau$ is natural.

□

Let $\iota : I \dot{\to} I$ be the identity natural transformation of the identity functor I of the category $\mathbf{B}$. The derivations

$$\begin{aligned}
(\iota \diamond \tau)_A &= ((I \lhd \tau) \bullet (\iota \rhd T))_A \\
&= (I \lhd \tau)_A \circ (\iota \rhd T)_A \\
&= I(\tau_A) \circ \iota_{TA} \\
&= \tau_A \circ 1_{TA} \\
&= \tau_A
\end{aligned}$$

and

$$\begin{aligned}
(\varrho \diamond \iota)_B &= ((\varrho \rhd I) \bullet (R \lhd \iota))_B \\
&= (\varrho \rhd I)_B \circ (R \lhd \iota)_B \\
&= \varrho_{IB} \circ R(\iota_B) \\
&= \varrho_B \circ R(1_B) \\
&= \varrho_B \circ 1_{RB} \\
&= \varrho_B
\end{aligned}$$

show that ι is the identity for the horizontal composition $\diamond$. As we already know, it is also an identity for the vertical composition $\bullet$.

This composition $\langle \varrho, \tau \rangle \mapsto \varrho \diamond \tau$ is readily shown to be associative. To prove this, we assume additional given functors $P, P' : \mathbf{C} \to \mathbf{D}$ together with a natural transformation $\pi : P \dot{\to} P'$. By definition of the horizontal composite we can write the equations

$$\begin{aligned}
\pi \diamond (\varrho \diamond \tau) &= (P' \lhd (\varrho \diamond \tau)) \bullet (\pi \rhd (R \circ T)) \\
&= (P' \lhd ((R' \lhd \tau) \bullet (\varrho \rhd T))) \bullet (\pi \rhd (R \circ T)).
\end{aligned}$$

Use the results from proposition 8 and the associativity of $\bullet$ to derive

$$\pi \diamond (\varrho \diamond \tau) = (P' \lhd (R' \lhd \tau)) \bullet ((P' \lhd (\varrho \rhd T)) \bullet (\pi \rhd (R \circ T))).$$

Proposition 6 entitles us to transform this into

$$\pi \diamond (\varrho \diamond \tau) \quad = \quad ((P' \circ R') \triangleleft \tau) \bullet (((P' \triangleleft \varrho) \triangleright T) \bullet ((\pi \triangleright R) \triangleright T)).$$

Now use proposition 8 again and read the definition of the horizontal composite backwards. This gives

$$\begin{aligned} \pi \diamond (\varrho \diamond \tau) &= ((P' \circ R') \triangleleft \tau) \bullet (((P' \triangleleft \varrho) \bullet (\pi \triangleright R)) \triangleright T) \\ &= ((P' \circ R') \triangleleft \tau) \bullet ((\pi \diamond \varrho) \triangleright T) \\ &= (\pi \diamond \varrho) \diamond \tau. \end{aligned}$$

Hence the horizontal composition $\diamond$ is associative.

Note that omitting the composition signs (except $\bullet$), as is customary in most text books about categories, the last associativity proof would have the form

$$\begin{aligned} \pi(\varrho\tau) &= P'(\varrho\tau) \bullet \pi(RT) \\ &= P'(R'\tau \bullet \varrho T) \bullet \pi RT \\ &= (P'R'\tau \bullet P'\varrho T) \bullet \pi RT \\ &= P'R'\tau \bullet (P'\varrho T \bullet \pi RT) \\ &= P'R'\tau \bullet (P'\varrho \bullet \pi R)T \\ &= (P'R')\tau \bullet (\pi\varrho)T \\ &= (\pi\varrho)\tau. \end{aligned}$$

This is more readable but it hides the *necessity* of our propositions 6 and 8.

5.4 Several properties

We now show that the use of right and left composites can always be avoided using horizontal composites and corresponding identity natural transformations instead. Use the notation of the last definition and for an arbitrary object A of $\mathbf{A}$ derive

$$\begin{aligned} (\iota_R \diamond \tau)_A &= (R \triangleleft \tau)_A \circ (\iota_R \triangleright T)_A \\ &= (R \triangleleft \tau)_A \circ (\iota_R)_{(TA)} \\ &= (R \triangleleft \tau)_A \circ \iota_{(R(TA))} \\ &= (R \triangleleft \tau)_A \circ 1_{R(TA)} \\ &= (R \triangleleft \tau)_A \end{aligned}$$

and

$$\begin{aligned} (\varrho \diamond \iota_T)_A &= (\varrho \triangleright T)_A \circ (R \triangleleft \iota_T)_A \\ &= (\varrho \triangleright T)_A \circ R((\iota_T)_A) \\ &= (\varrho \triangleright T)_A \circ R(\iota_{(TA)}) \\ &= (\varrho \triangleright T)_A \circ R(1_{(TA)}) \\ &= (\varrho \triangleright T)_A \circ 1_{R(TA)}) \\ &= (\varrho \triangleright T)_A. \end{aligned}$$

Hence it follows $\varrho \diamond \iota_T = \varrho \triangleright T$ and $\iota_R \diamond \tau = R \triangleleft \tau$.

Therefore (16) can be rewritten

$$\begin{aligned} \varrho \diamond \tau &= (\varrho \diamond \iota_{T'}) \bullet (\iota_R \diamond \tau) \\ &= (\iota_{R'} \diamond \tau) \bullet (\varrho \diamond \iota_T) \end{aligned} \tag{17}$$

If we transform the equations in proposition 6, we recognize that they can be interpreted as special instances of the associativity law for $\diamond$.

Note that the composition sign $\diamond$ in most books is often omitted, i.e., $\varrho\tau$ denotes $\varrho \diamond \tau$. Usually, also $\tau \triangleright R$ and $R \triangleleft \tau$ are simply written as τR and $R\tau$ respectively. For the last expressions we could write $\tau \diamond \iota_R = \tau\iota_R$ and $\iota_R \diamond \tau = \iota_R\tau$ which would only omit the $\diamond$ sign but not identify the functor to be its own identity natural transformation.

In this paper, however, we shall always write out the composition signs. Although this makes formulae longer, it avoids ambiguity in all cases.

As stated in [ML71] there is a special relation (*interchange law*) involving the compositions $\diamond$ and $\bullet$.

Proposition 10 (Interchange Law) *For functors* $T, T', T'' : \mathbf{A} \to \mathbf{B}$ *and* $R, R', R'' : \mathbf{B} \to \mathbf{C}$ *and natural transformations* $\tau : T \dot{\to} T'$, $\tau' : T' \dot{\to} T''$, $\varrho : R \dot{\to} R'$ *and* $\varrho' : R' \dot{\to} R''$ *following identity holds:*

$$(\varrho' \bullet \varrho) \diamond (\tau' \bullet \tau) = (\varrho' \diamond \tau') \bullet (\varrho \diamond \tau) \tag{18}$$

Proof: This law can readily be shown valid using the definition of the horizontal composite in terms of right and left composites together with the associativity of $\bullet$ and our proposition 8. The derivation sequence is

$$\begin{aligned} (\varrho' \bullet \varrho) \diamond (\tau' \bullet \tau) &\overset{(16)}{=} (R'' \triangleleft (\tau' \bullet \tau)) \bullet ((\varrho' \bullet \varrho) \triangleright T) \\ &\overset{(15)}{=} (R'' \triangleleft \tau') \bullet (R'' \triangleleft \tau) \bullet (\varrho' \triangleright T) \bullet (\varrho \triangleright T) \\ &\overset{(16)}{=} (R'' \triangleleft \tau') \bullet (\varrho' \diamond \tau) \bullet (\varrho \triangleright T) \\ &\overset{(16)}{=} (R'' \triangleleft \tau') \bullet (\varrho' \triangleright T') \bullet (R' \triangleleft \tau) \bullet (\varrho \triangleright T) \\ &\overset{(16)}{=} (\varrho' \diamond \tau') \bullet (\varrho \diamond \tau). \end{aligned}$$

$\square$

As stated above, the right and left composites can be avoided. Apply that to the equations (15) in proposition 8 and recall that ι_T and ι_P respectively are identities for the vertical composition. Now we could obtain these equations using the interchange law.

$$\begin{aligned} (\varrho' \bullet \varrho) \triangleright T &= (\varrho' \bullet \varrho) \diamond \iota_T \\ &= (\varrho' \bullet \varrho) \diamond (\iota_T \bullet \iota_T) \\ &= (\varrho' \diamond \iota_T) \bullet (\varrho \diamond \iota_T) \\ &= (\varrho' \triangleright T) \bullet (\varrho \triangleright T) \end{aligned}$$

$$
\begin{aligned}
P \triangleleft (\varrho' \bullet \varrho) &= \iota_P \diamond (\varrho' \bullet \varrho) \\
&= (\iota_P \bullet \iota_P) \diamond (\varrho' \bullet \varrho) \\
&= (\iota_P \diamond \varrho') \bullet (\iota_P \diamond \varrho) \\
&= (P \triangleleft \varrho') \bullet (P \triangleleft \varrho)
\end{aligned}
$$

However, we used the proposition 8 to prove the interchange law, and so it would be wrong to call the above derivation a proof for the proposition 8.

Since ι_T, $\iota_{T'}$, ι_R and $\iota_{R'}$ are identities for the composition $\bullet$, equations (17) become a special instance of the interchange law.

There is another useful property which we shall use for our further investigations.

Let $T : \mathbf{A} \to \mathbf{B}$ and $R : \mathbf{B} \to \mathbf{C}$.
Then we have

$$
\begin{aligned}
(\iota_{R \circ T})_A &= 1_{R(TA)} \\
&= R(1_{TA}) \\
&= (R \triangleleft \iota_T)_A \\
&= (\iota_R \diamond \iota_T)_A.
\end{aligned}
$$

Hence $\iota_{R \circ T} = \iota_R \diamond \iota_T$.

Since in the following sections identity natural transformations of a functor to itself will occur frequently, we follow the convention that the corresponding Greek letter of a functor (name) denotes its identity natural transformation, for example, $\sigma = \iota_S$, $\tau = \iota_T$, $\iota = \iota_I$, $\sigma' = \iota_{S'}$, $\hat{\sigma} = \iota_{\hat{S}}$, ... This is almost the same as to let a functor symbol denote its own identity natural transformation.

However, in the following discussion we use some special transformations, denoted by η, μ, ε and including their primed versions, which are not identity natural transformations of some functor.

Look at the definition of a monad. With this new notation the equations (1) and (2) can now rewritten as

$$
\mu \bullet (\sigma \diamond \mu) = \mu \bullet (\mu \diamond \sigma) \tag{19}
$$

and

$$
\mu \bullet (\sigma \diamond \eta) = \mu \bullet (\eta \diamond \sigma) \; = \; \sigma. \tag{20}
$$

6 Counter example

Look back to the example in section 4. Use the meaning of the symbols there and derive

$$\begin{aligned}(\eta \diamond \eta')_A\, a\, x &= (\eta_{S'A} \circ \eta'_A)\, a\, x \\ &= \eta_{S'A}(\eta'_A\, a)\, x \\ &= \eta_{S'A}(i_A(a))\, x \\ &= i_A(a).\end{aligned}$$

We recognize $\hat{\eta} = \eta \diamond \eta'$.

It is tempting to suppose that $\hat{\mu} = \mu \diamond \mu'$, but this cannot be so, since

$$\mu \diamond \mu' : S \circ S \circ S' \circ S' \dot{\to} S \circ S',$$

but we require

$$\hat{\mu} : S \circ S' \circ S \circ S' \dot{\to} S \circ S'$$

and $S \circ S \circ S' \circ S' \neq S \circ S' \circ S \circ S'$ in general. Even if $S = S'$, $\mu \diamond \mu'$ need not to be a natural transformation which satisfies the properties of a monad.

To see this, consider in **Set** the monad $\mathcal{M}$ from section 4 and try to build a monad $\hat{\mathcal{M}} = \langle S \circ S, \eta \diamond \eta, \mu \diamond \mu \rangle$. The proof of associativity of the multiplication $\mu \diamond \mu$ will fail as the following deduction shows.

Write $\hat{\mu}$ for $\mu \diamond \mu$. First we show how $\hat{\mu}$ behaves. By definition of μ and $\diamond$, we have for any object A, any arrow $f \in (((A^X)^X)^X)^X$ and $x, x' \in X$

$$\begin{aligned}\hat{\mu}_A\, f\, x\, x' &= (\mu \diamond \mu)_A\, f\, x\, x' \\ &= ((S \lhd \mu) \bullet (\mu \rhd S^2))_A\, f\, x\, x' \\ &= ((S \lhd \mu)_A \circ (\mu \rhd S^2)_A)\, f\, x\, x' \\ &= (S(\mu_A) \circ \mu_{S^2 A})\, f\, x\, x' \\ &= (\mu_A)_* \,(\mu_{S^2 A}\, f)\, x\, x' \\ &= (\mu_A \circ (\mu_{S^2 A}\, f))\, x\, x' \\ &= \mu_A\, (\mu_{S^2 A}\, f\, x)\, x' \\ &= \mu_A\, (f\, x\, x)\, x' \\ &= (f\, x\, x)\, x'\, x'.\end{aligned} \tag{21}$$

To show the associativity examine $\hat{\mu}_A \circ S^2(\hat{\mu}_A)$ which is the corresponding expression for the left-hand side of (1) in the definition of a monad.

$$\begin{aligned}(\hat{\mu}_A \circ S^2(\hat{\mu}_A))\, f\, x\, x' &= (\hat{\mu}_A \circ (\hat{\mu}_A)_{**})\, f\, x\, x' \\ &= \hat{\mu}_A((\hat{\mu}_A)_{**}\, f)\, x\, x' \\ &\overset{(21)}{=} (\hat{\mu}_A)_{**}\, f\, x\, x\, x'\, x' \\ &= ((\hat{\mu}_A)_* \circ f)\, x\, x\, x'\, x' \\ &= (\hat{\mu}_A)_*\, (f\, x)\, x\, x'\, x' \\ &= (\hat{\mu}_A \circ (f\, x))\, x\, x'\, x' \\ &= \hat{\mu}_A\, (f\, x\, x)\, x'\, x' \\ &\overset{(21)}{=} (f\, x\, x)\, x'\, x'\, x'\, x'\end{aligned}$$

Now consider the right-hand side.

$$\begin{array}{rcl}(\hat{\mu}_A \circ \hat{\mu}_{S^2A})\, f\, x\, x' & = & \hat{\mu}_A\, (\hat{\mu}_{S^2A}\, f)\, x\, x' \\ & \overset{(21)}{=} & (\hat{\mu}_{S^2A}\, f)\, x\, x\, x'\, x' \\ & = & (\hat{\mu}_{S^2A}\, f\, x\, x)\, x'\, x' \\ & \overset{(21)}{=} & (f\, x\, x\, x\, x)\, x'\, x'\end{array}$$

Obviously, both results are different. Hence $\hat{\mathcal{M}} = \langle S \circ S, \eta \diamond \eta, \mu \diamond \mu \rangle$ is not a monad.

One could ask where this comes from. A deeper look into the structure of $\hat{\mu}$ could give a possible answer. Nevertheless, this counter example, which seems to have a simple structure of the involved functor and natural transformation, suggests that there is no way to use μ and μ' to construct $\hat{\mu}$ (at least if no further structure is known).

At this stage we accept this, but it seems unnatural that the multiplication $\hat{\mu}$ of a composed monad can independently build up without making use of μ and μ'. Instead of searching for ways to construct $\hat{\mu}$ from μ and μ', we consider the structure of the domain and the codomain of $\hat{\mu}$ and observe two possible ways to reduce $S \circ S' \circ S \circ S'$ to $S \circ S'$.

1. Let $\hat{\varepsilon} : S' \circ S \dot{\to} I$ be a natural transformation. In this case define $\hat{\mu} = \sigma \diamond \hat{\varepsilon} \diamond \sigma'$.

2. Given two natural transformations $\varepsilon : S \dot{\to} I$ and $\varepsilon' : S' \dot{\to} I$. Then let $\hat{\mu} = (\sigma \diamond \varepsilon) \diamond (\varepsilon' \diamond \sigma')$

Since horizontal composition is associative the second way is a special case of the first.

It seems unlikely that we can find such natural transformations ε and ε', and even to find a natural transformation like $\hat{\varepsilon}$ can be an unsolvable problem.

For example, take the functors S and S' to be the same as in section 4. Now, choose $X = \{0\}$, $Y = \{y\}$ and $A = B = \{0, 1\}$. As A is isomorphic to $SA = A^X$ it remains to show that no natural transformation $\hat{\varepsilon} : S' \dot{\to} I$ can be found. This can be proved by looking at the diagram

$$\begin{tikzcd} A+Y \arrow[r, "\hat{\varepsilon}_A"] \arrow[d, "f+1_Y"'] & A \arrow[d, "f"] \\ A+Y \arrow[r, "\hat{\varepsilon}_A"] & A \end{tikzcd}$$

for the function $f : A \to A$ where $f(a) = 1 - a$. For $j_A(y) \in A + Y$ take the upper and right-hand arrow and derive

$$\begin{array}{rcl}(f \circ \hat{\varepsilon}_A)(j_A(y)) & = & f(\hat{\varepsilon}_A(j_A(y))) \\ & = & 1 - \hat{\varepsilon}_A(j_A(y)),\end{array}$$

and for the left-hand and lower arrow we obtain

$$\begin{array}{rcl}(\hat{\varepsilon}_A \circ (f + 1_Y))(j_A(y)) & = & \hat{\varepsilon}_A((f + 1_Y)(j_A(y))) \\ & = & \hat{\varepsilon}_A(j_A(y)).\end{array}$$

Hence we conclude that no function $\hat{\varepsilon}_A$ with the property to be a component of a natural transformation can be defined, and therefore $\hat{\varepsilon} : S' \circ S \dot{\to} I$ cannot exist.

Despite this fact, we suppose a given natural transformation $\hat{\varepsilon} : S' \circ S \dot{\to} I$ in addition to our monads $\mathcal{M}$ and $\mathcal{M}'$ and examine the question whether or not this could lead to a monad construction. As we shall see, we can give a positive answer if we add two further assumptions.

7 Composition of two monads

This section presents the result that under certain circumstances one part of our question can be answered. The answer does not deal with the general case. Since both functor components of the given monads are endofunctors and in this section we try to construct a monad which has as functor component the composite of these functors, the given functors must be composable. Therefore the given monads are here required to be in the same category.

We already assumed to have a natural transformation $\hat{\varepsilon} : S' \circ S \dot{\rightarrow} I$ in addition to two given monads in the same category. In order to give further conditions for these data to form a monad in a certain way, we use an important concept in category theory, namely adjunctions. It is a general result that every adjunction gives rise to a monad. Therefore we first define adjunctions and then impose constraints to the given monads that their functor parts become a pair of adjoint functors. Finally, we construct a 'composed' monad and give a proof for that.

7.1 Definition of adjunctions

For our purpose it is more convenient, not to copy the definition of adjunctions in [ML71] but to use an equivalent one.

Definition 9 *Given two categories* $\mathbf{A}$*,* $\mathbf{B}$*. An* adjunction *from* $\mathbf{A}$ *to* $\mathbf{B}$ *is a quadruple* $\langle T, R, \eta, \varepsilon \rangle : \mathbf{A} \rightharpoonup \mathbf{B}$*, where* $T : \mathbf{A} \to \mathbf{B}$ *and* $R : \mathbf{B} \to \mathbf{A}$ *are functors, and* $\eta : I \dot{\rightarrow} R \circ T$*,* $\varepsilon : T \circ R \dot{\rightarrow} J$ *are natural transformations such that*

$$\iota_R = (\iota_R \diamond \varepsilon) \bullet (\eta \diamond \iota_R) \tag{22}$$

$$\iota_T = (\varepsilon \diamond \iota_T) \bullet (\iota_T \diamond \eta) \tag{23}$$

hold. Here $I : \mathbf{A} \to \mathbf{A}$ *is the identity functor of* $\mathbf{A}$*, and* $J : \mathbf{B} \to \mathbf{B}$ *the identity functor of* $\mathbf{B}$*.* η *is called the* unit *and* ε *the* counit *of the adjunction. The functor* T *is called* left adjoint *to* R*, and* R *is named* right adjoint *to* T*.*

7.2 Given monads and constraints

In order to build a monad using adjunctions, we start with the important fact that from every adjunction a monad can be obtained.

Proposition 11 *Given an adjunction* $\langle T, R, \eta, \varepsilon \rangle : \mathbf{A} \rightharpoonup \mathbf{B}$*. Then the structure* $\mathcal{M} = \langle S, \eta, \mu \rangle$ *where* $S = R \circ T$ *and* $\mu = \varrho \diamond \varepsilon \diamond \tau$ *is a monad in* $\mathbf{A}$*.*

Recall $\varrho = \iota_R$ and $\tau = \iota_T$.
Proof: Let $\check{S} = T \circ R$. Our proof consists of two parts. First we proof the associativity property and then the unit property for a monad.

(Associativity property)
Since ε is natural the following diagram commutes.

$$\begin{tikzcd} \check{S} \arrow[d, "\varepsilon"] \\ I \end{tikzcd} \qquad \begin{tikzcd} \check{S} \circ \check{S} \arrow[r, "\varepsilon \diamond \check{\sigma}"] \arrow[d, "\check{\sigma} \diamond \varepsilon"] & I \circ \check{S} \arrow[d, "\iota \diamond \varepsilon"] \\ \check{S} \circ I \arrow[r, "\varepsilon \diamond \iota"] & I \circ I \end{tikzcd}$$

We now can lift this diagram twice. At first from the right with T and then from the left with R. Since R and T are functors the resulting diagram still commutes. This is

$$\begin{tikzcd} R \circ \check{S} \circ T \arrow[d, "\varrho \diamond \varepsilon \diamond \tau"] \\ R \circ I \circ T \end{tikzcd} \qquad \begin{tikzcd} R \circ \check{S} \circ \check{S} \circ T \arrow[r, "\varrho \diamond \varepsilon \diamond \check{\sigma} \diamond \tau"] \arrow[d, "\varrho \diamond \check{\sigma} \diamond \varepsilon \diamond \tau"] & R \circ I \circ \check{S} \circ T \arrow[d, "\varrho \diamond \iota \diamond \varepsilon \diamond \tau"] \\ R \circ \check{S} \circ I \circ T \arrow[r, "\varrho \diamond \varepsilon \diamond \iota \diamond \tau"] & R \circ I \circ I \circ T \end{tikzcd}$$

Using the identities

$$\begin{aligned} S &= R \circ T \\ \check{S} &= T \circ R \\ \sigma &= \iota_S = \iota_{(R \circ T)} = \iota_R \diamond \iota_T = \varrho \diamond \tau \\ \check{\sigma} &= \iota_{\check{S}} = \iota_{(T \circ R)} = \iota_T \diamond \iota_R = \tau \diamond \varrho \\ \mu &= \varrho \diamond \varepsilon \diamond \tau \end{aligned}$$

and the fact that ι is the identity for $\diamond$ and I for $\circ$ the last diagram simplifies to

$$\begin{tikzcd} S^3 \arrow[r, "\mu \diamond \sigma"] \arrow[d, "\sigma \diamond \mu"] & S^2 \arrow[d, "\mu"] \\ S^2 \arrow[r, "\mu"] & S \end{tikzcd}$$

The commutativity of the last diagram proves the property of μ to be a multiplication of a monad.

(Unit property)
Observe that in the last part we did neither use (22) nor (23). However, to prove η to have the properties which renders $\mathcal{M}$ a monad, we need these conditions. Equations (22) and (23) give rise to the commutativity of

$$\begin{tikzcd} R = I \circ R \arrow[r, "\eta \diamond \varrho"] \arrow[dr, "\varrho"'] & R \circ T \circ R \arrow[d, "\varrho \diamond \varepsilon"] \\ & R = R \circ I \end{tikzcd}$$

and

$$\begin{tikzcd}
T \circ R \circ T \arrow[d, "\varepsilon \diamond \tau"] & & T \circ I = T \arrow[ll, "\tau\diamond\eta"'] \arrow[dll, "\tau"] \\
I \circ T = T & &
\end{tikzcd}$$

Lifting the first diagram with T on the right and the second with R on the left, and then composing both diagrams together, yields

$$\begin{tikzcd}
R \circ T \arrow[r, "\eta\diamond\varrho\diamond\tau"] \arrow[dr, "\varrho \diamond \tau"'] & R \circ T \circ R \circ T \arrow[d, "\varrho \diamond \varepsilon \diamond \tau"] & R \circ T \arrow[l, "\varrho\diamond\tau\diamond\eta"'] \arrow[dl, "\varrho \diamond \tau"] \\
& R \circ T &
\end{tikzcd}$$

which simplifies to

$$\begin{tikzcd}
S \arrow[r, "\eta\diamond\sigma"] \arrow[dr, "\sigma"'] & S^2 \arrow[d, "\mu"] & S \arrow[l, "\sigma\diamond\eta"'] \arrow[dl, "\sigma"] \\
& S &
\end{tikzcd}$$

and is still commutative, since R and T are functors. The last diagram, however, proves the identity property of η in the monad $\mathcal{M}$.

Thus η and μ have their necessary properties, so that $\mathcal{M}$ is indeed a monad.

$\Box$

Using this proposition enables us to give the following fact.

Corollary 1 *Let $\mathcal{M} = \langle S, \eta, \mu \rangle$ and $\mathcal{M}' = \langle S', \eta', \mu' \rangle$ be two monads in an arbitrary category $\mathbf{A}$. In addition let $\hat{\varepsilon} : S' \circ S \dot{\to} I$ be a natural transformation from $S' \circ S$ to the identity functor I on $\mathbf{A}$. Under the conditions*

$$\sigma = (\sigma \diamond \hat{\varepsilon}) \bullet (\hat{\eta} \diamond \sigma) \tag{24}$$

$$\sigma' = (\hat{\varepsilon} \diamond \sigma') \bullet (\sigma' \diamond \hat{\eta}) \tag{25}$$

the structure $\hat{\mathcal{M}} = \langle \hat{S}, \hat{\eta}, \hat{\mu} \rangle$ where $\hat{S} = S \circ S'$, $\hat{\eta} = \eta \diamond \eta'$ and $\hat{\mu} = \sigma \diamond \hat{\varepsilon} \diamond \sigma'$, is a monad.

Recall that $\sigma = \iota_S$ and $\sigma' = \iota_{S'}$.
Proof: S and S' are endofunctors of $\mathbf{A}$, and as a result of the properties of the horizontal composite $\hat{\eta} = \eta \diamond \eta' : S \circ S' \dot{\to} I$ is a natural transformation. The given conditions (24) and (25) are exactly the relations (22) and (23) respectively. Therefore we can conclude that $\langle S', S, \hat{\eta}, \hat{\varepsilon} \rangle : \mathbf{A} \rightharpoonup \mathbf{A}$ is an adjunction. Use the last proposition, and the corollary is thus shown to be valid.

$\Box$

Thus we can readily construct a monad if it can be shown that the functor parts of the given monads are a pair of adjoint functors and the corresponding natural transformations are known.

This result is obviously not surprising and it also makes no use of the given monad multiplications. However, it emphasizes the importance of adjunctions in category theory, and it can also give a way for further research. For example, let $S = S'$ and consider the construction in the category which is the full image of the given category $\mathbf{A}$ under the functor S, i.e., it has objects SA for A in $\mathbf{A}$ and arrows $Sf : SA \to SA'$ for arrows $f : A \to A'$ in $\mathbf{A}$. In this category the multiplication μ can be regarded as a natural transformation $\varepsilon : S \dot{\to} I$ and thus we find a relation between μ and a former unknown ε. For this construction it would be useful to introduce the concept of (in [ML71] called) T-algebras to form an adjunction from a monad which in turn enables us to reconstruct this monad in the standard way given above. See also [Par70] for constructing an adjunction from a monad. Whether or not the transition to a subcategory is too restrictive, cannot yet be answered. This might depend upon the monads involved.

8 Conclusion

This paper treated a small area of category theory to investigate the question whether or not there is a canonical way to compose two monads. This question could not be answered in general. We looked at a particular example of a composed monad in section 4, mentioned also an example where it is impossible to construct a monad in a way one would think to be canonical, and at the end of this dissertation we pointed a direction in which further research could be done to obtain a more acceptable result than here is given.

The author already looked for ways to build the natural transformations ε and ε' from the given monads $\mathcal{M}$ and $\mathcal{M}'$ without assuming a mapping $\hat{\varepsilon}$. This, however, leads us to form the new monad in another category or at least in a subcategory of the given one. As this opens our restricted question, only to consider one category, and even from the new point of view there are still problems, these results are not presented here.

Thus further investigation could use following facts:

- Every monad can be build from an adjunction and there is a way to find an adjunction with such properties.
- Adjunctions can be composed (see [ML71], IV.8.).

Perhaps the direction, we have given, could yield an acceptable algorithm to build a new monad, but, of course, it cannot cover all possible cases because it makes no use of further internal structure of the given monads. Once the relation of monads and adjunctions have been studied, it could be useful to look back at the example, we have given, to investigate which internal structure there enables us to build a new monad although in general there is no natural transformation $\hat{\varepsilon} : S' \circ S \dot{\rightarrow} I$ as this dissertation has shown.

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